Hypergraph Partitioning using Tensor Eigenvalue Decomposition

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Objective

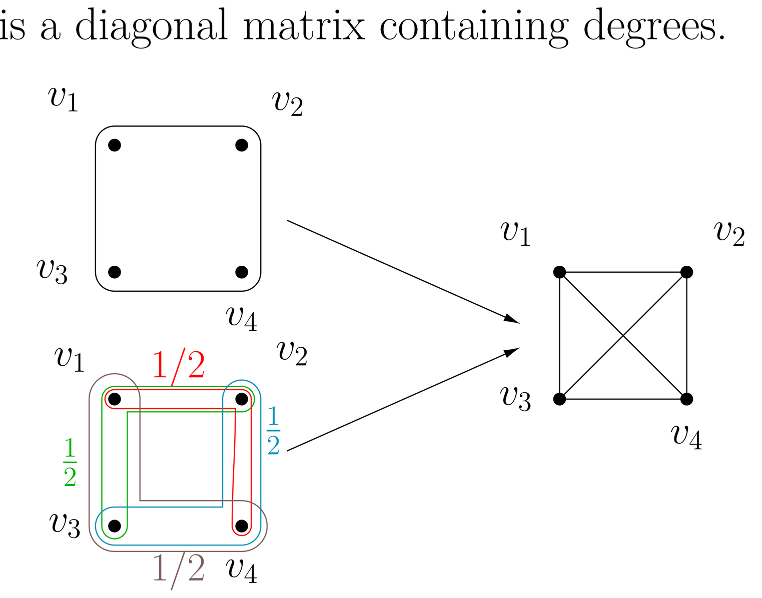
For **k-uniform undirected weighted hypergraph** $G = (V, E)$, remove a subset of ∂E hyperedges, such that resulting partitions have minimum ratio-cut value.

Hyperedge Reduction

A hypergraph $G = (V, E)$ can be represented by vertex-edge incidence matrix $\mathbf{H}$ of dimension $|V| \times |E|$ whose entry $h(i, j) = 1$ if $v_i \in e_j$ and 0 otherwise. The adjacency matrix for reduced hypergraph using clique expansion:

$$\mathbf{A}_r = \mathbf{H}\mathbf{W}\mathbf{H}^T - \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix containing degrees.



Multiple hypergraphs may reduce to same graph.

Hypergraph Representation

A natural representation of hypergraphs is a k-order n-dimensional tensor $\mathcal{A}$ [1], which consists of n^k entries:

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} w_{e_j} \frac{1}{(k-1)!} & \text{if } (i_1, i_2, \ldots, i_k) = \{e_j\} \quad e_j \in E \\ 0 & \text{otherwise} \end{cases}$$

It should be noted that $\mathcal{A}$ is a "*super-symmetric*" tensor. The degree of a vertex v_i is given by

$$d(v_i) = \sum_{i_k=1}^{n} \ldots \sum_{i_3=1}^{n} \sum_{i_2=1}^{n} a_{i i_2 i_3 \ldots i_k}$$

The Laplacian tensor $\mathcal{L}$ is defined as:

$$\mathcal{L} = \mathcal{D} - \mathcal{A}$$

Spectral decomposition [2] using

$$\mathcal{L}\mathbf{x}^{k-1} = \lambda \mathbf{x}, \qquad \mathbf{x}^T\mathbf{x} = 1$$

where $(\lambda, \mathbf{x}) \in (\mathbb{R}, \mathbb{R}^n \backslash \{0\}^n)$ satifying above is called the Z-eigenpair and $\mathcal{L}\mathbf{x}^{k-1} \in \mathbb{R}^n$, whose i^{th} component is defined

$$\left[\mathcal{L}\mathbf{x}^{k-1}\right]_i = \sum_{i_k=1}^{n} \ldots \sum_{i_3=1}^{n} \sum_{i_2=1}^{n} l_{i i_2 i_3 \ldots i_k} x_{i_2} x_{i_3} \ldots x_{i_k}$$

Relaxation of min Ratio-cut

For disjoint partitions C_i and $\bar{C}_i$:

$$\text{cut}(C_i, \bar{C}_i) = \sum_{e_j \in \partial E} w_{e_j} |C_i \cap e_j|$$

$$\min_{C_1, \ldots, C_p} \sum_{i=1}^{p} \frac{\text{cut}(C_i, \bar{C}_i)}{k|C_i|^{k/2}}$$

Equivalent to:

$$\min_{\mathbf{f}_1, \ldots, \mathbf{f}_p} \sum_{i=1}^{p} \mathcal{L}\mathbf{f}_i^k$$

$$\mathcal{L} = \mathcal{D} - \mathcal{A}, \qquad f_{i,j} = \begin{cases} \frac{1}{\sqrt{|C_j|}} & v_i \in C_j \\ 0 & \text{otherwise} \end{cases}$$

Relaxation:

$$\min_{\{\mathbf{f}_1, \ldots, \mathbf{f}_p\} \in \mathbb{R}^{|V|}} \sum_{i=1}^{p} \mathcal{L}\mathbf{f}_i^k, \quad \text{s.t} \quad \mathbf{f}_i^T\mathbf{f}_i = 1$$

Solution: $\mathcal{L}\mathbf{f}^{k-1} = \lambda \mathbf{f}$

Construction Cost

Theorem: *The hypergraph Laplacian objective function for a k-uniform hypergraph can be expressed as*

$$\mathcal{L}\mathbf{x}^k = \sum_{e_j \in E} l_{e_j}(\mathbf{x})$$

$$l_{e_j}(\mathbf{x}) = w_{e_j} \left(\sum_{i_k \in e_j} x_{i_k}^k - k \prod_{i_k \in e_j} x_{i_k} \right)$$

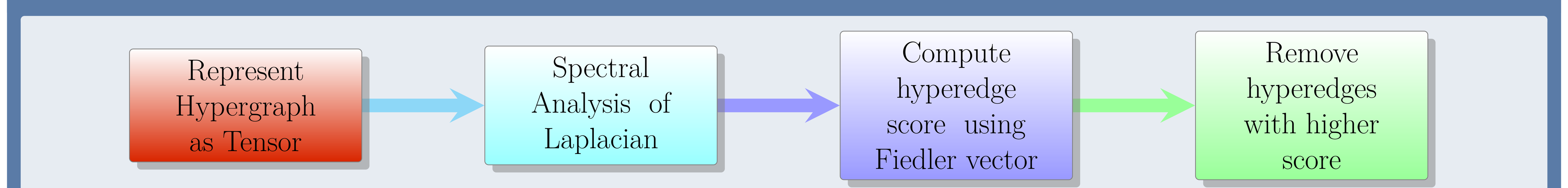
$$= w_{e_j} k \left(\text{A.M}_{i_j \in e_j} \left(x_{i_k}^k\right) - \text{G.M}_{i_j \in e_j} \left(|x_{i_k}|^k\right)(-1)^{n_s} \right)$$

where $n_s = |\{i_j : x_{i_j} < 0\}|$, A.M and G.M stand for the arithmetic and geometric means, respectively.

Example: Consider a hypergraph $G = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{\{1, 2, 3\}\}$. The score corresponding to the hyperedge $\{1, 2, 3\}$ is given by:

$$l_{e_j}(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$$

Proposed Algorithm



Spectral Analysis

Tensor eigenvalue decomposition arises from:

$$\min_{\mathbf{x}} \quad \mathcal{L}\mathbf{x}^k = \sum_{i_k=1}^{n} \ldots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} l_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}$$

such that $\mathbf{x}^T\mathbf{x} = 1$

The eigenvector with minimum positive λ satisfying above equation is termed as Fiedler eigenvector and can be computed by following optimization problem

$$\mathbf{v}_\star = \underset{\mathbf{x}}{\text{argmin}} \quad \mathcal{L}\mathbf{x}^k > 0, \qquad \text{s. t} \quad \mathbf{x}^T\mathbf{x} = 1$$

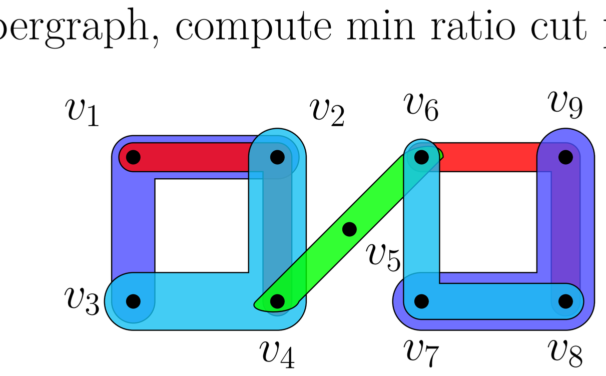
The corresponding eigenvalue can be computed as $\lambda_\star = \mathcal{L}\mathbf{v}_\star^k$.

Challenges

- Eigenvectors may not be orthogonal for symmetric tensors.
- Odd order tensor have negative eigenvalues.

Ex 1: 3-uniform hypergraph

For given hypergraph, compute min ratio cut partitions.



$\mathbf{f}_{11}$	$\mathbf{f}_{21}$	$\mathbf{f}_{31}$	$\mathbf{f}_{41}$
-0.05	0.06	0.47	0.47
0.03	0.03	0.46	0.46
0.06	-0.05	0.47	0.47
0.23	0.23	0.42	0.42
0.34	0.34	0.34	0.34
0.42	0.42	0.23	0.23
0.47	0.47	-0.05	0.06
0.46	0.46	0.03	0.03
0.47	0.47	0.06	-0.05

Fiedler Eigenvectors

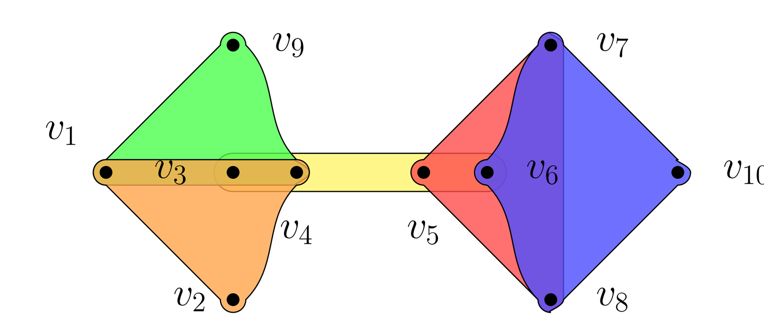
hyperedges	$l_{e_j}(\mathbf{f}_{11})$	$l_{e_j}(\mathbf{f}_{21})$	$l_{e_j}(\mathbf{f}_{31})$	$l_{e_j}(\mathbf{f}_{41})$
$\{1, 2, 3\}$	0.0004	0.0004	0	0
$\{1, 2, 4\}$	0.0127	0.0111	0.0025	0.0025
$\{2, 3, 4\}$	0.0111	0.0127	0.0025	0.0025
$\{4, 5, 6\}$	0.0278	0.0278	0.0278	0.0278
$\{6, 7, 8\}$	0.0025	0.0025	0.0127	0.0111
$\{7, 8, 9\}$	0	0	0.0004	0.0004
$\{6, 8, 9\}$	0.0025	0.0025	0.0111	0.0127

Hyperedge Score

Remove hyperedge $\{v_4, v_5, v_6\}$ (higher score)

Ex 2: 4-uniform hypergraph

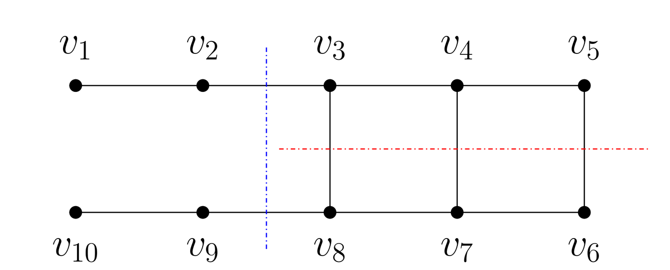
Consider the 4-uniform hypergraph



Remove the hyperedge $\{v_3, v_4, v_5, v_6\}$ (higher score)

hyperedges	$l_{e_j}(\mathbf{x})$
$\{1, 3, 4, 9\}$	0.0355
$\{1, 2, 3, 4\}$	0.0355
$\{3, 4, 5, 6\}$	0.3785
$\{5, 6, 7, 8\}$	0.3670
$\{6, 7, 8, 10\}$	0.1577

Ex 3: 2-uniform hypergraph



Ratio-cut $= \frac{3}{5} + \frac{3}{5} = 1.2$, Existing Method

Ratio-cut $= \frac{2}{4} + \frac{2}{6} = 0.83$, Proposed Approach (Optimal)

Conclusions & Future Work

1. Partitioning of hypergraphs without reduction is proposed. Also demonstrated the improvement for graphs.
2. Scalability of tensor eigenvalue decomposition to apply on real datasets.
3. Theoretical analysis of the proposed algorithm.

References

[1] A. Banerjee, A. Char, and B. Mondal, "Spectra of general hypergraphs," *Linear Algebra and its Applications*, vol. 518, pp. 14–30, 2017.

[2] L. Qi and Z. Luo, *Tensor analysis: spectral theory and special tensors*, vol. 151, Siam 2017.

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