Identification of Output-Error (OE) Models using QZ Algorithm

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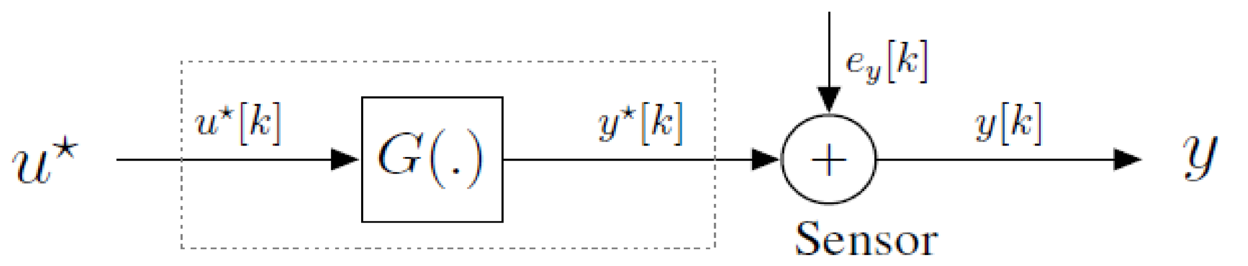
Introduction & Objectives

Identify the **dynamic difference equation (DE)** model,

$$y^\star[k] + \sum_{i=1}^{n_a} a_i y^\star[k-i] = \sum_{j=D}^{n_b} b_j u^\star[k-j]$$

from output **measurements** and **known** inputs, $y[k] = y^\star[k] + e_y[k]$ and $u[k] = u^\star[k]$, as shown below and $e_y \sim \mathcal{N}(0, \sigma^2_{e_u})$



Framework / Assumptions :

- Linear time-invariant **(LTI)** processes with unknown **order** and **delay**
- **Noise variance** of the output is unknown

State of the art methods

1. **Prediction Error Minimization**
   - Methodology is to minimize n-step ahead prediction error
2. **Steiglitz-McBride Algorithm**
   - Attempts to minimize the mean-square error between system and model outputs
3. **Subspcace based methods**
   - Utilizes combination of projection & linear algebra based approach

All these approaches deliver unbiased and consistent parameter estimates but assume to have knowledge of order and delay a priori.

QZ Algorithm

The QZ algorithm is numerical method for solving generalized eigenvalue problem

$$\mathbf{Av} = \mathbf{\Lambda B v} \quad (1)$$

without performing matrix inversion $\mathbf{B}$.

1. The idea is to transform eqn. 1 with unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$
$$\mathbf{QAZy} = \mathbf{\Lambda QBZy}, \qquad \text{with} \quad \mathbf{v} = \mathbf{Zy} \quad (2)$$
   such that $\mathbf{QAZ}$ and $\mathbf{QBZ}$ are upper triangular
2. Eigenvalues can be computed from the diagonals of the triangular form. Eigenvectors can be computed by transforming back the eigenvectors of triangular problem with $\mathbf{Z}$

Problem Formulation

Define the matrix of lagged measurements as

$$\mathbf{Z}_L = \left[\mathbf{z_L}[L] \ \ \mathbf{z_L}[L+1] \ \cdots \ \mathbf{z_L}[N]\right]^T; \qquad \mathbf{z_L}[k] = \left[y[k] \ \ y[k-1] \ \cdots \ y[k-L] \ \ u[k] \ \ u[k-1] \ \cdots u[k-L]\right]^T$$

The original problem of system identification can be solved to identify the linear relation among the lagged variables of $\mathbf{Z}_L$ from Eqn. 1 by choosing

$$\mathbf{A} \triangleq \frac{1}{N}\mathbf{Z}_L^T\mathbf{Z}_L$$

and $\mathbf{B}$ as a diagonal matrix with entries as $[\mathbf{1}_{L+1} \quad \mathbf{0}_{L+1}]$

Motivating Example

A 2^{nd} order unit-delay process, excited with a white PRBS input (N = 1023) :

$$y^\star[k] + 0.4y^\star[k-1] + 0.6y^\star[k-2] = 1.2u^\star[k-1]$$

Measurements $y[k] = y^\star[k] + e_y[k]$, such that (i)$e_y \sim \mathcal{N}(0, 0.244)$ (ii)SNR = 10.

- Assume order and delay are known. Construct matrix of lagged variables:
$$\mathbf{Z}_2 = \left[\mathbf{z_2}[2] \ \ \mathbf{z_2}[3] \ \cdots \ \mathbf{z_2}[N]\right]^T; \qquad \mathbf{z_2}[k] = \left[y[k] \ \ y[k-1] \ \ y[k-2] \ \ u[k-1]\right]^T$$
- Applying QZ algorithm on $\mathbf{\Sigma_Z}$ with $\mathbf{B} = \text{diag}([1 \quad 1 \quad 1 \quad 0])$ delivers :
$$\Lambda = \left[\infty \ 3.52 \ 2.91 \ 0.2408\right]; \qquad \bar{\mathbf{v}} = \left[0.825 \ 0.332 \ 0.503 \ -1\right]$$
- Estimated DE model (from last eigenvector, $\bar{\mathbf{v}}$):
$$\hat{y}[k] + 0.402\hat{y}[k-1] + 0.609\hat{y}[k-2] = 1.211u^\star[k-1] \quad ✓$$

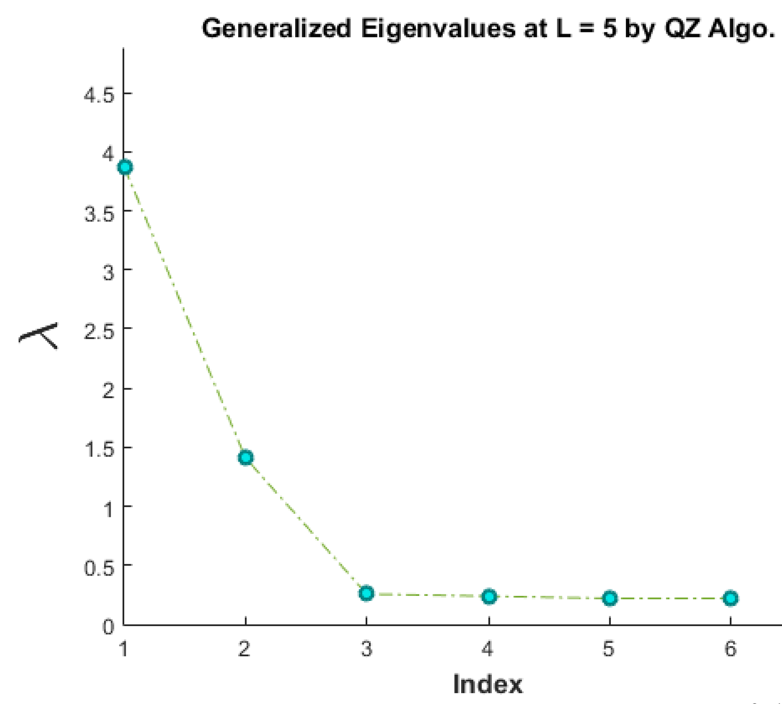
QZ algorithm delivers unbiased estimates with known delay and order.

Proposed Algorithm

1. Construct the data matrix $\mathbf{Z}_L$ by stacking the data upto a sufficiently large lag L and perform the generalized eigenvalue decomposition of the covariance matrix via QZ algorithm by choosing $\mathbf{B} = \text{diag}[\mathbf{1}_{L\times 1} \quad \mathbf{0}_{L\times 1}]$.
2. Identify the true number of linear relations, d^* and the order of the process using the relation $\hat{\eta} = L - d^* + 1 \triangleq \max(n_a, n_b)$.
3. Re-configure the data matrix $\mathbf{Z}_{\hat{\eta}}$ by stacking only upto the estimated order $\hat{\eta}$, followed by an eigenvalue decomposition of the covariance matrix to obtain the model from eigenvector corresponding to minimum eigenvalue.
4. Perform a statistical tests of significance on estimated parameters.

Motivating Example: Revisited with Proposed Algo.

As the order and delay is assumed to be unknown, construct $\mathbf{Z}_L$ with $L = 5$



Estimated order :

$\hat{\eta} = L - \hat{d} + 1 = 5 - 4 + 1 = 2$ ✓

Estimated difference equation :

$$y[k] + \underset{\pm(0.0346)}{0.3996}\, y[k-1] + \underset{\pm(0.016)}{0.6004} y[k-2] = \underset{\pm(0.036)}{0.001}\, u[k] + \underset{\pm(0.035)}{1.2}\, u[k-1] - \underset{\pm(0.087)}{0.066}\, u[k-2]$$

Insights from confidence interval:

$\hat{n}_a = 2, \ \hat{n}_b = 1, \ \hat{D} = 1$ ✓

Estimated difference equation with 95% CIs via MC simulations of 200 runs

$$y[k] + \underset{\pm(0.0179)}{0.3999}\, y[k-1] + \underset{\pm(0.0172)}{0.5999}\, y[k-2] = \underset{\pm(0.0377)}{1.1994}\, u[k-1] \quad ✓$$

Order Estimation Comparison with State Space Algo.

A two-tank system is considered to show the supremacy of proposed algorithm over state space algorithm for order estimation.

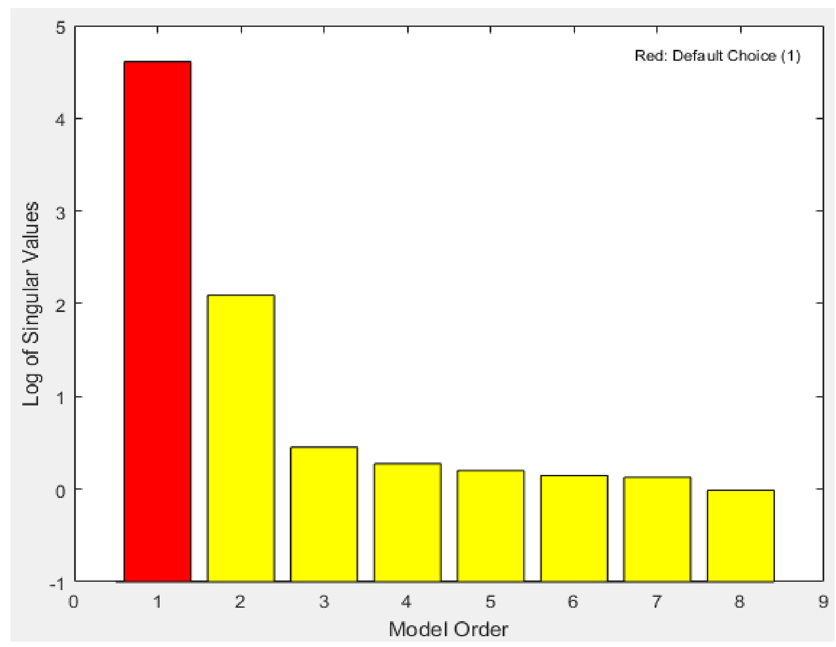


Figure: N4SID: $\hat{\eta} = 1$ ×

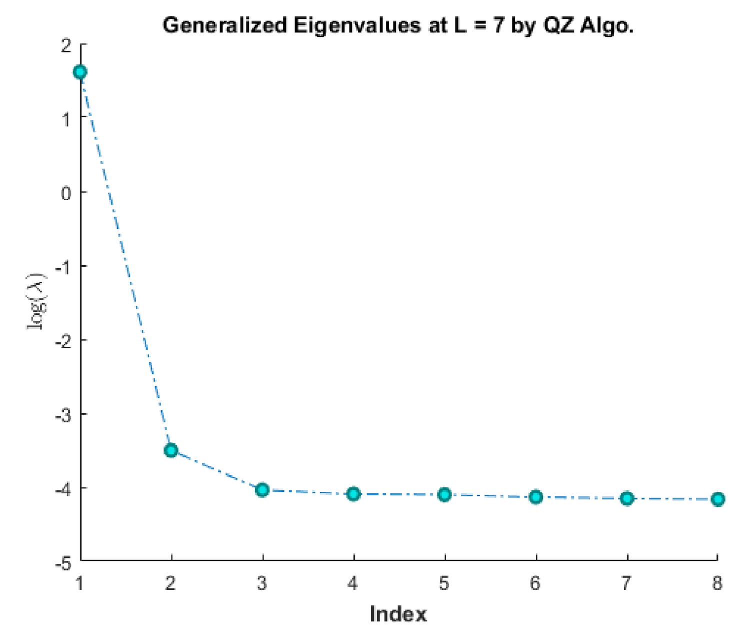


Figure: QZ: $\hat{\eta} = 7 - 6 + 1 = 2$ ✓

Order is estimated correctly by the proposed algorithm whereas the heuristic approach by state space algorithm fails

Conclusions

- The efficacy of the proposed algorithm in identifying the model in an automated manner with no prior knowledge about the process has been demonstrated
- Further study involves extension of the proposed algorithm to model multi-input, multi-output (MIMO) systems

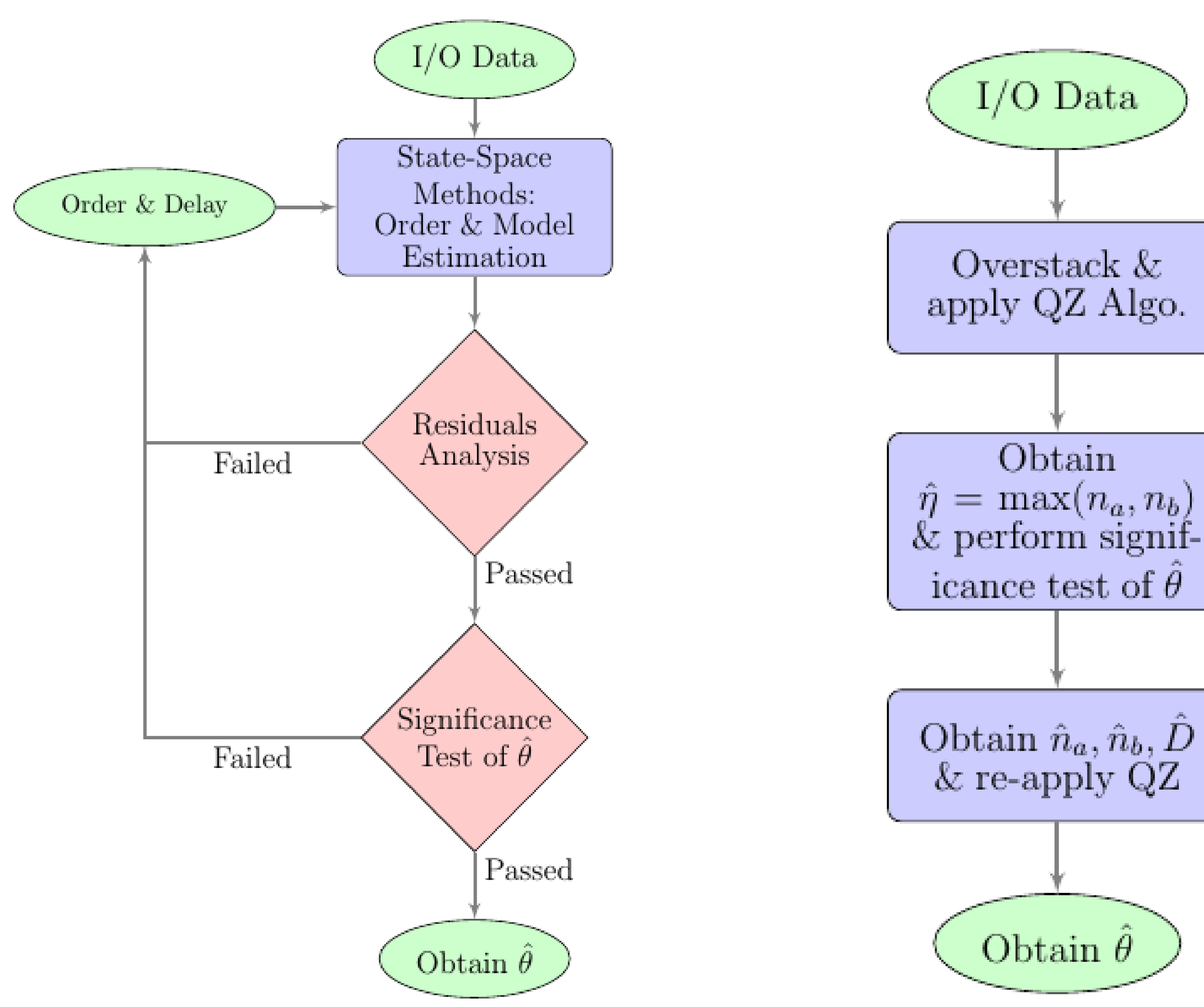


Figure: Comparison of PEM & QZ

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